

COMMON FIXED POINTS VIA λ -SEQUENCES IN G -METRIC SPACES.YAÉ OLATOUNDI GABA^{1,2,*}

ABSTRACT. In this article, we use λ -sequences to derive common fixed points for a family of self-mappings defined on a complete G -metric space. We imitate some existing techniques in our proofs and show that the tools emlyed can be used at a larger scale. These results generalize well known results in the literature.

1. INTRODUCTION AND PRELIMINARIES

The generalization of the Banach contraction mapping principle has been a heavily investigated branch of research. In recent years, several authors have obtained *fixed* and *common fixed point* results for various classes of mappings in the setting of many generalized metric spaces. One of them, the G -metric space, is our focus in this paper and fixed point results, in this setting, presented by authors like Abbas[1], Gaba[2, 4], Mustafa[7], Vetro[8] and many more, are enlighting on the subject. Moreover, in [3], we introduced the concept of λ -sequence which extended the idea of α -series proposed by Vetro et al. in [8]. The present article exclusively presents natural extensions of some results already given by Abbas[1] and Vetro[8], and therefore generalizes some recent results regarding fixed point theory in G -metric spaces. We also show how the idea of λ -sequence are used in proving some of these results. The method builds on the convergence of an appropriate series of coefficients. We also make use of a special class of homogeneous functions. Recent and similar work can also be read in [2, 3, 4, 5].

We recall here some key results that will be useful in the rest of this manuscript. The basic concepts and notations attached to the idea of G -metric spaces can be read extensively in [7] but for the convenience of the reader, we discuss the most important ones.

Definition 1.1. (Compare [7, Definition 3]) Let X be a nonempty set, and let the function $G : X \times X \times X \rightarrow [0, \infty)$ satisfy the following properties:

- (G1) $G(x, y, z) = 0$ if $x = y = z$ whenever $x, y, z \in X$;
- (G2) $G(x, x, y) > 0$ whenever $x, y \in X$ with $x \neq y$;
- (G3) $G(x, x, y) \leq G(x, y, z)$ whenever $x, y, z \in X$ with $z \neq y$;
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$, (symmetry in all three variables);
- (G5)

$$G(x, y, z) \leq [G(x, a, a) + G(a, y, z)]$$

for any points $x, y, z, a \in X$.

Then (X, G) is called a **G -metric space**.

The property (G3) is crucial and shall play a key role in our proofs.

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Proposition 1.2. (Compare [7, Proposition 6]) Let (X, G) be a G -metric space. Then for a sequence $(x_n) \subseteq X$, the following are equivalent

- (i) (x_n) is G -convergent to $x \in X$.
- (ii) $\lim_{n,m \rightarrow \infty} G(x, x_n, x_m) = 0$.
- (iii) $\lim_{n \rightarrow \infty} G(x, x_n, x_n) = 0$.
- (iv) $\lim_{n \rightarrow \infty} G(x_n, x, x) = 0$.

Proposition 1.3. (Compare [7, Proposition 9]) In a G -metric space (X, G) , the following are equivalent

- (i) The sequence $(x_n) \subseteq X$ is G -Cauchy.
- (ii) For each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq N$.

Definition 1.4. (Compare [7, Definition 9]) A G -metric space (X, G) is said to be complete if every G -Cauchy sequence in (X, G) is G -convergent in (X, G) .

Definition 1.5. (Compare [7, Definition 4]) A G -metric space (X, G) is said to be symmetric if

$$G(x, y, y) = G(x, x, y), \text{ for all } x, y \in X.$$

Definition 1.6. (Compare [4, Definition 2.1]) A sequence $(x_n)_{n \geq 1}$ in a metric space (X, d) is a λ -sequence if there exist $0 < \lambda < 1$ and $n(\lambda) \in \mathbb{N}$ such that

$$\sum_{i=1}^{L-1} d(x_i, x_{i+1}) \leq \lambda L \text{ for each } L \geq n(\lambda) + 1.$$

Definition 1.7. (Compare [2, Definition 6]) A sequence $(x_n)_{n \geq 1}$ in a G -metric space (X, G) is a λ -sequence if there exist $0 < \lambda < 1$ and $n(\lambda) \in \mathbb{N}$ such that

$$\sum_{i=1}^{L-1} G(x_i, x_{i+1}, x_{i+1}) \leq \lambda L \text{ for each } L \geq n(\lambda) + 1.$$

Definition 1.8. (Compare [8, Definition 2.1]) For a sequence $(a_n)_{n \geq 1}$ of nonnegative real numbers, the series $\sum_{n=1}^{\infty} a_n$ is an α -series if there exist $0 < \lambda < 1$ and $n(\lambda) \in \mathbb{N}$ such that

$$\sum_{i=1}^L a_i \leq \lambda L \text{ for each } L \geq n(\lambda).$$

Remark 1.9. For a given λ -sequence $(x_n)_{n \geq 1}$ in a G -metric space (X, d) , the sequence $(\beta_n)_{n \geq 1}$ of nonnegative real numbers defined by

$$\beta_i = d(x_i, x_{i+1}, x_{i+1}),$$

is an α -series.

Moreover, any non-increasing λ -sequence of elements of \mathbb{R}^+ endowed with the \max ¹ metric is also an α -series. Therefore, λ -sequences generalise α -series but to ease computations, we shall consider, throughout the paper, α -series².

¹The \max metric m refers to $m(x, y) = \max\{x, y\}$

²However, the reader can convince himself that using λ -sequences do not add to the complexity of the problem.

2. FIRST GENERALIZATIONS RESULTS

We begin with the following generalisation of [8, Theorem 2.1], the main result of Vetro et al.

Let Φ be the class of continuous, non-decreasing, sub-additive and homogeneous functions $F : [0, \infty) \rightarrow [0, \infty)$ such that $F^{-1}(0) = \{0\}$.

Theorem 2.1. *Let (X, G) be a complete G -metric space and $\{T_n\}$ be a family of self mappings on X such that*

$$\begin{aligned} F(G(T_i x, T_j y, T_k z)) \leq & F\left({}_k\Theta_{i,j} \left[G(x, T_i x, T_i x) + \frac{1}{2}[(G(y, T_j y, T_j y) + G(z, T_k z, T_k z))] \right] \right) \\ & + F({}_k\Delta_{i,j} G(x, y, z)) \end{aligned} \quad (2.1)$$

for all $x, y, z \in X$ with $x \neq y$, $0 \leq {}_k\Theta_{i,j}, {}_k\Delta_{i,j} < 1$; $i, j, k = 1, 2, \dots$, and some $F \in \Phi$, homogeneous with degree s . If

$$\sum_{i=1}^{\infty} \left[\frac{[({}_{i+2}\Theta_{i,i+1})^s + ({}_{i+2}\Delta_{i,i+1})^s]}{1 - ({}_{i+2}\Theta_{i,i+1})^s} \right]$$

is an α -series, then $\{T_n\}$ have a unique common fixed point in X .

Proof. We will proceed in two main steps.

Claim 1: $\{T_n\}_{n \geq 1}$ have a common fixed point in X .

For any $x_0 \in X$, we construct the sequence (x_n) by setting $x_n = T_n(x_{n-1})$, $n = 1, 2, \dots$.

We assume without loss of generality that $x_m \neq x_n$ for all $n \neq m \in \mathbb{N}$. Using (2.1), we obtain, for the triplet (x_0, x_1, x_2) ,

$$\begin{aligned} F(G(x_1, x_2, x_3)) &= F(G(T_1 x_0, T_2 x_1, T_3 x_2)) \\ &\leq ({}_3\Theta_{1,2})^s F\left(\left[G(x_0, x_1, x_1) + \frac{1}{2}G(x_1, x_2, x_2) + \frac{1}{2}G(x_2, x_3, x_3)\right]\right) \\ &\quad + ({}_3\Delta_{1,2})^s F(G(x_0, x_1, x_2)). \end{aligned}$$

By property (G3) of G , one knows that

$$G(x_i, x_{i+1}, x_{i+1}) \leq G(x_{i-1}, x_i, x_{i+1}) \quad \text{and} \quad G(x_i, x_i, x_{i+1}) \leq G(x_i, x_{i+1}, x_{i+2}).$$

Hence,

$$\begin{aligned}
F(G(x_1, x_2, x_3)) &= F(G(T_1x_0, T_2x_1, T_3x_2)) \\
&\leq ({}_3\Theta_{1,2})^s F\left(\left[G(x_0, x_1, x_2) + \frac{1}{2}G(x_1, x_2, x_3) + \frac{1}{2}G(x_1, x_2, x_3)\right]\right) \\
&\quad + ({}_3\Delta_{1,2})^s F(G(x_0, x_1, x_2)) \\
&= ({}_3\Theta_{1,2})^s F(G(x_1, x_2, x_3)) + [({}_3\Theta_{1,2})^s + ({}_3\Delta_{1,2})^s] F(G(x_1, x_2, x_0))
\end{aligned}$$

i.e.

$$F(G(x_1, x_2, x_3)) \leq \frac{[({}_3\Theta_{1,2})^s + ({}_3\Delta_{1,2})^s]}{1 - ({}_3\Theta_{1,2})^s} F(G(x_0, x_1, x_2)).$$

Also we get

$$\begin{aligned}
F(G(x_2, x_3, x_4)) &\leq \frac{[({}_4\Theta_{2,3})^s + ({}_4\Delta_{2,3})^s]}{1 - ({}_4\Theta_{2,3})^s} F(G(x_1, x_2, x_3)) \\
&\leq \left[\frac{[({}_4\Theta_{2,3})^s + ({}_4\Delta_{2,3})^s]}{1 - ({}_4\Theta_{2,3})^s} \right] \left[\frac{[({}_3\Theta_{1,2})^s + ({}_3\Delta_{1,2})^s]}{1 - ({}_3\Theta_{1,2})^s} \right] F(G(x_0, x_1, x_2)).
\end{aligned}$$

Repeating the above reasoning, we obtain

$$F(G(x_n, x_{n+1}, x_{n+2})) \leq \prod_{i=1}^n \left[\frac{[(i+2)\Theta_{i,i+1}]^s + [(i+2)\Delta_{i,i+1}]^s}{1 - (i+2)\Theta_{i,i+1}^s} \right] F(G(x_0, x_1, x_2)).$$

If we set

$$r_i = \left[\frac{[(i+2)\Theta_{i,i+1}]^s + [(i+2)\Delta_{i,i+1}]^s}{1 - (i+2)\Theta_{i,i+1}^s} \right],$$

we have that

$$F(G(x_n, x_{n+1}, x_{n+2})) \leq \left[\prod_{i=1}^n r_i \right] F(G(x_0, x_1, x_2)).$$

Therefore, for all $l > m > n > 2$

$$\begin{aligned}
G(x_n, x_m, x_l) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + \cdots + G(x_{l-1}, x_{l-1}, x_l) \\
&\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) \\
&\quad + \cdots + G(x_{l-2}, x_{l-1}, x_l).
\end{aligned}$$

Using the fact that F is sub-additive, we write

$$\begin{aligned}
F(G(x_n, x_m, x_l)) &\leq \left(\left[\prod_{i=1}^n r_i \right] + \left[\prod_{i=1}^{n+1} r_i \right] + \cdots + \left[\prod_{i=1}^{l-2} r_i \right] \right) F(G(x_0, x_1, x_2)) \\
&= \sum_{k=0}^{l-n-2} \left[\prod_{i=1}^{n+k} r_i \right] F(G(x_0, x_1, x_2)) \\
&= \sum_{k=n}^{l-2} \left[\prod_{i=1}^k r_i \right] F(G(x_0, x_1, x_2)).
\end{aligned}$$

Now, let λ and $n(\lambda)$ as in Definition 1.8, then for $n \geq n(\lambda)$ and using the fact that the geometric mean of non-negative real numbers is at most their arithmetic mean, it follows that

$$\begin{aligned}
F(G(x_n, x_m, x_l)) &\leq \sum_{k=n}^{l-2} \left[\frac{1}{k} \left(\sum_{i=1}^k r_i \right) \right]^k F(G(x_0, x_1, x_2)) \\
&= \left(\sum_{k=n}^{l-2} \alpha^k \right) F(G(x_0, x_1, x_2)) \\
&\leq \frac{\alpha^n}{1-\alpha} F(G(x_0, x_1, x_2)).
\end{aligned}$$

As $n \rightarrow \infty$, we deduce that $G(x_n, x_m, x_l) \rightarrow 0$. Thus (x_n) is a G -Cauchy sequence. and since X is complete there exists $u \in X$ such that (x_n) G -converges to u .

Moreover, for any positive integers k, l , we have

$$\begin{aligned}
F(G(x_n, T_k u, T_l u)) &= F(G(T_n x_{n-1}, T_k u, T_l u)) \\
&\leq F \left(({}_l \Theta_{n,k}) \left[G(x_{n-1}, T_n x_{n-1}, T_n x_{n-1}) + \frac{1}{2} [(G(u, T_k u, T_k u) + G(u, T_l u, T_l u))] \right] \right) \\
&\quad + F(({}_l \Delta_{n,k}) G(x_{n-1}, u, u)).
\end{aligned}$$

Letting $n \rightarrow \infty$, and using property (G3) we obtain

$$F(G(u, T_k u, T_l u)) \leq ({}_l \Theta_{n,k})^s F(G(u, T_k u, T_l u)),$$

and this is a contradiction, unless $u = T_k u = T_l u$, since ${}_l \Theta_{n,k} < 1$. Then u is a common fixed point of $\{T_n\}$.

Claim 2: u is the unique common fixed point of $\{T_m\}$.

Finally, we prove the uniqueness of the common fixed point u . To this aim, let us suppose that v is another common fixed point of $\{T_m\}$, that is, $T_m(v) = v$, $\forall m \geq 1$. Then, using (2.1) again, we have

$$F(G(u, v, v)) = F(G(T_m u, T_m v, T_m v)) \leq ({}_m \Delta_{m,m})^s F(G(u, v, v)),$$

which yields $u = v$, since ${}_m \Delta_{m,m} < 1$. So, u is the unique common fixed point of $\{T_m\}$. \square

Theorem 2.2. Let (X, G) be a complete G -metric space and $\{T_n\}$ be a family of self mappings on X such that

$$F(G(T_i^p x, T_j^p y, T_k^p z)) \leq F\left(({}_k\Theta_{i,j}) \left[G(x, T_i^p x, T_i^p x) + \frac{1}{2}[(G(y, T_j^p y, T_j^p y) + G(z, T_k^p z, T_k^p z))] \right] \right) + F(({}_k\Delta_{i,j})G(x, y, z)) \quad (2.2)$$

for all $x, y, z \in X$ with $x \neq y$, $0 \leq {}_k\Theta_{i,j}, {}_k\Delta_{i,j} < 1$; $i, j, k = 1, 2, \dots$, some positive integer p , and some $F \in \Phi$, homogeneous with degree s . If

$$\sum_{i=1}^{\infty} \left[\frac{[(i+2)\Theta_{i,i+1}]^s + (i+2)\Delta_{i,i+1}^s}{1 - (i+2)\Theta_{i,i+1}^s} \right]$$

is an α -series, then $\{T_n\}$ have a unique common fixed point in X .

Proof. It follows from Theorem 2.1, that the family $\{T_n^p\}$ have a unique common fixed point x^* . Now for any positive integers $i, j, i \neq j$,

$$T_i(x^*) = T_i T_i^p(x^*) = T_i^p T_i(x^*) \text{ and } T_j(x^*) = T_j T_j^p(x^*) = T_j^p T_j(x^*),$$

i.e. $T_i(x^*)$ and $T_j(x^*)$ are also fixed points for T_i^p and T_j^p ³. Since the common fixed point of $\{T_n^p\}$ is unique, we deduce that

$$x^* = T_i(x^*) = T_j(x^*) \text{ for all } i.$$

□

The next result, corollary of Theorem 2.1, corresponds to the result presented by Vetro [8, Theorem 2.1].

Corollary 2.3. (Compare [8, Theorem 2.1]) Let (X, G) be a complete G -metric space and $\{T_n\}$ be a family of self mappings on X such that

$$G(T_i x, T_j y, T_k z) \leq ({}_k\Theta_{i,j})[G(x, T_i x, T_i x) + (G(y, T_j y, T_j y))] + ({}_k\Delta_{i,j})G(x, y, z) \quad (2.3)$$

for all $x, y, z \in X$ with $x \neq y$, $0 \leq {}_k\Theta_{i,j}, {}_k\Delta_{i,j} < 1$, $i, j, k = 1, 2, \dots$. If

$$\sum_{i=1}^{\infty} \left[\frac{[(i+2)\Theta_{i,i+1}] + (i+2)\Delta_{i,i+1}}{1 - (i+2)\Theta_{i,i+1}} \right]$$

is an α -series, then $\{T_n\}$ have a unique common fixed point in X .

Proof. In Theorem 2.1, take $F = Id_{[0,\infty)}$ ⁴, $j = k$ and $y = z$.

□

³Remember that any fixed point of T_i^p is a fixed point of T_j^p for $i \neq j$, Cf. Theorem 2.1.

⁴The identity map on $[0, \infty)$

3. SECOND GENERALIZATIONS RESULTS

The next generalisation is that of [1, Theorem 2.1], the main result of Abbas et al. Instead of considering three maps, we consider a family of maps like in the previous case. Moreover, to show the reader that λ -sequences do not add to the complexity of the problem, we shall use them in the next statement.

Theorem 3.1. *Let X be a complete G -metric space (X, G) and $\{T_n\}$ be a sequence of self mappings on X . Assume that there exist three sequences (a_n) , (b_n) and (c_n) of elements of X such that*

$$\begin{aligned} G(T_i x, T_j y, T_k z) &\leq ({}_k\Delta_{i,j})G(x, y, z) + ({}_k\Theta_{i,j})[G(T_i x, x, x) + G(y, T_j y, y) + G(z, z, T_k z)] \\ &\quad + ({}_k\Lambda_{i,j})[G(T_i x, y, z) + G(x, T_j y, z) + G(x, y, T_k z)], \end{aligned} \quad (3.1)$$

for all $x, y, z \in X$ with $0 \leq {}_k\Delta_{i,j} + 3({}_k\Theta_{i,j}) + 4({}_k\Lambda_{i,j}) < 1/2$, $i, j, k = 1, 2, \dots$, where ${}_k\Delta_{i,j} = G(a_i, a_j, a_k)$, ${}_k\Theta_{i,j} = G(b_i, b_j, b_k)$ and ${}_k\Lambda_{i,j} = G(c_i, c_j, c_k)$. If the sequence (r_i) where

$$r_i = \left[\frac{[({}_{i+2}\Delta_{i,i+1}) + 2({}_{i+2}\Theta_{i,i+1}) + 3({}_{i+2}\Lambda_{i,i+1})]}{1 - ({}_{i+2}\Theta_{i,i+1}) - ({}_{i+2}\Lambda_{i,i+1})} \right]$$

is a non-increasing λ -sequence of \mathbb{R}^+ endowed with the max⁵ metric, then $\{T_n\}$ have a unique common fixed point in X . Moreover, any fixed point of T_i is a fixed point of T_j for $i \neq j$.

Proof. We will proceed in two main steps.

Claim 1: Any fixed point of T_i is also a fixed point of T_j and T_k for $i \neq j \neq k \neq i$.

Assume that x^* is a fixed point of T_i and suppose that $T_j x^* \neq x^*$ and $T_k x^* \neq x^*$. Then

$$\begin{aligned} G(x^*, T_j x^*, T_k x^*) &= G(T_i x^*, T_j x^*, T_k x^*) \\ &\leq ({}_k\Delta_{i,j})G(x^*, x^*, x^*) + ({}_k\Theta_{i,j})[G(T_i x^*, x^*, x^*) + G(x^*, T_j x^*, x^*) + G(x^*, x^*, T_k x^*)] \\ &\quad + ({}_k\Lambda_{i,j})[G(T_i x^*, x^*, x^*) + G(x^*, T_j x^*, x^*) + G(x^*, x^*, T_k x^*)] \\ &\leq [({}_k\Theta_{i,j}) + ({}_k\Lambda_{i,j})][G(x^*, T_j x^*, T_k x^*) + G(x^*, T_j x^*, T_k x^*)] \\ &\leq [(2{}_k\Theta_{i,j}) + (2{}_k\Lambda_{i,j})][G(x^*, T_j x^*, T_k x^*)], \end{aligned}$$

which is a contradiction unless $T_i x^* = x^* = T_j x^* = T_k x^*$.

Claim 2:

For any $x_0 \in X$, we construct the sequence (x_n) by setting $x_n = T_n(x_{n-1})$, $n = 1, 2, \dots$. We assume without loss of generality that $x_n \neq x_m$ for all $n \neq m$. Using (3.1), we obtain

$$\begin{aligned} G(x_1, x_2, x_3) &= G(T_1 x_0, T_2 x_1, T_3 x_2) \\ &\leq ({}_3\Delta_{1,2})G(x_0, x_1, x_2) + ({}_3\Theta_{1,2})[G(x_1, x_0, x_0) + G(x_1, x_2, x_1) + G(x_2, x_2, x_3)] \\ &\quad + ({}_3\Lambda_{1,2})[G(x_1, x_1, x_2) + G(x_0, x_2, x_2) + G(x_0, x_1, x_3)]. \end{aligned}$$

By property (G3), one can write

⁵The max metric m refers to $m(x, y) = \max\{x, y\}$

$$\begin{aligned}
G(x_1, x_2, x_3) &= G(T_1x_0, T_2x_1, T_3x_2) \\
&\leq ({}_3\Delta_{1,2})G(x_0, x_1, x_2) + ({}_3\Theta_{1,2})[G(x_1, x_0, x_2) + G(x_1, x_2, x_0) + G(x_1, x_2, x_3)] \\
&\quad + ({}_3\Lambda_{1,2})[G(x_1, x_0, x_2) + G(x_0, x_1, x_2) + G(x_0, x_1, x_3)]
\end{aligned}$$

Again since

$$G(x_0, x_1, x_3) \leq G(x_0, x_2, x_2) + G(x_2, x_1, x_3) \leq G(x_0, x_1, x_2) + G(x_2, x_1, x_3),$$

we obtain,

$$\begin{aligned}
G(x_1, x_2, x_3) &= G(T_1x_0, T_2x_1, T_3x_2) \\
&\leq ({}_3\Delta_{1,2})G(x_0, x_1, x_2) + ({}_3\Theta_{1,2})[G(x_1, x_0, x_2) + G(x_1, x_2, x_0) + G(x_1, x_2, x_3)] \\
&\quad + ({}_3\Lambda_{1,2})[G(x_1, x_0, x_2) + G(x_0, x_1, x_2) + G(x_0, x_1, x_2) + G(x_2, x_1, x_3)],
\end{aligned}$$

that is

$$[1 - ({}_3\Theta_{1,2}) - ({}_3\Lambda_{1,2})]G(x_1, x_2, x_3) \leq [({}_3\Delta_{1,2}) + 2({}_3\Theta_{1,2}) + 3({}_3\Lambda_{1,2})]G(x_0, x_1, x_2).$$

Hence

$$G(x_1, x_2, x_3) \leq \frac{[({}_3\Delta_{1,2}) + 2({}_3\Theta_{1,2}) + 3({}_3\Lambda_{1,2})]}{1 - ({}_3\Theta_{1,2}) - ({}_3\Lambda_{1,2})}G(x_0, x_1, x_2).$$

Also we get

$$\begin{aligned}
G(x_2, x_3, x_4) &\leq \frac{[({}_4\Delta_{2,3}) + 2({}_4\Theta_{2,3}) + 3({}_4\Lambda_{2,3})]}{1 - ({}_4\Theta_{2,3}) - ({}_4\Lambda_{2,3})}G(x_1, x_2, x_3) \\
&\leq \left[\frac{[({}_4\Delta_{2,3}) + 2({}_4\Theta_{2,3}) + 3({}_4\Lambda_{2,3})]}{1 - ({}_4\Theta_{2,3}) - ({}_4\Lambda_{2,3})} \right] \left[\frac{[({}_3\Delta_{1,2}) + 2({}_3\Theta_{1,2}) + 3({}_3\Lambda_{1,2})]}{1 - ({}_3\Theta_{1,2}) - ({}_3\Lambda_{1,2})} \right] G(x_0, x_1, x_2).
\end{aligned}$$

Repeating the above reasoning, we obtain

$$G(x_n, x_{n+1}, x_{n+2}) \leq \prod_{i=1}^n \left[\frac{[({}_{i+2}\Delta_{i,i+1}) + 2({}_{i+2}\Theta_{i,i+1}) + 3({}_{i+2}\Lambda_{i,i+1})]}{1 - ({}_{i+2}\Theta_{i,i+1}) - ({}_{i+2}\Lambda_{i,i+1})} \right] G(x_0, x_1, x_2)$$

If we set

$$r_i = \left[\frac{[({}_{i+2}\Delta_{i,i+1}) + 2({}_{i+2}\Theta_{i,i+1}) + 3({}_{i+2}\Lambda_{i,i+1})]}{1 - ({}_{i+2}\Theta_{i,i+1}) - ({}_{i+2}\Lambda_{i,i+1})} \right],$$

we have that

$$G(x_n, x_{n+1}, x_{n+2}) \leq \left[\prod_{i=1}^n r_i \right] G(x_0, x_1, x_2).$$

Therefore, for all $l > m > n > 2$

$$\begin{aligned}
G(x_n, x_m, x_l) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + \cdots + G(x_{l-1}, x_{l-1}, x_l) \\
&\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) \\
&\quad + \cdots + G(x_{l-2}, x_{l-1}, x_l),
\end{aligned}$$

and

$$\begin{aligned}
G(x_n, x_m, x_l) &\leq \left(\left[\prod_{i=1}^n r_i \right] + \left[\prod_{i=1}^{n+1} r_i \right] + \cdots + \left[\prod_{i=1}^{l-2} r_i \right] \right) G(x_0, x_1, x_2) \\
&= \sum_{k=0}^{l-n-2} \left[\prod_{i=1}^{n+k} r_i \right] G(x_0, x_1, x_2) \\
&= \sum_{k=n}^{l-2} \left[\prod_{i=1}^k r_i \right] G(x_0, x_1, x_2).
\end{aligned}$$

Now, let λ and $n(\lambda)$ as in Definition 1.8, then for $n \geq n(\lambda)$ and using the fact that the geometric mean of non-negative real numbers is at most their arithmetic mean, it follows that

$$\begin{aligned}
G(x_n, x_m, x_l) &\leq \sum_{k=n}^{l-2} \left[\frac{1}{k} \left(\sum_{i=1}^k r_i \right) \right]^k G(x_0, x_1, x_2) \\
&= \left(\sum_{k=n}^{l-2} \alpha^k \right) G(x_0, x_1, x_2) \\
&\leq \frac{\alpha^n}{1-\alpha} G(x_0, x_1, x_2).
\end{aligned}$$

As $n \rightarrow \infty$, we deduce that $G(x_n, x_m, x_l) \rightarrow 0$. Thus (x_n) is a G -Cauchy sequence. Moreover, since X is complete there exists $u \in X$ such that (x_n) G -converges to u .

If there exists n_0 such that $T_{n_0}u = u$, then by the claim 1, the proof of existence is complete. Otherwise for any positive integers k, l , we have

$$\begin{aligned}
G(x_n, T_k u, T_l u) &= G(T_n x_{n-1}, T_k u, T_l u) \\
&\leq ({}_l \Delta_{n,k}) G(x_{n-1}, u, u) + ({}_l \Theta_{n,k}) [G(T_n x_{n-1}, x_{n-1}, x_{n-1}) + G(u, T_k u, u) + G(u, u, T_l u)] \\
&\quad + ({}_l \Lambda_{n,k}) [G(T_n x_{n-1}, u, u) + G(x_{n-1}, T_k u, u) + G(x_{n-1}, u, T_l u)]
\end{aligned}$$

Letting $n \rightarrow \infty$, and using property (G3) we obtain

$$\begin{aligned}
G(u, T_k u, T_l u) &\leq ({}_l \Theta_{n,k}) [G(u, T_k u, u) + G(u, u, T_l u)] \\
&\quad + ({}_l \Lambda_{n,k}) [G(u, T_k u, u) + G(u, u, T_l u)] \\
&\leq [(2_k \Theta_{i,j}) + (2_k \Lambda_{i,j})] [G(u, T_k u, T_l u) + G(u, T_k u, T_l u)]
\end{aligned}$$

and this is a contradiction, unless $u = T_k u = T_l u$.

Finally, we prove the uniqueness of the common fixed point u . To this aim, let us suppose that v is another common fixed point of T_m , that is, $T_m(v) = v$, $\forall m \geq 1$. Then, using 3.1, we have

$$G(u, v, v) = G(T_n u, T_k v, T_l v) \leq ({}_l \Delta_{n,k})G(u, v, v) + 3({}_l \Lambda_{n,k})G(u, v, v),$$

which yields $u = v$. So, u is the unique common fixed point of $\{T_m\}$. \square

Following the same lines of the proof of Theorem 2.2, one can prove the next theorem.

Theorem 3.2. *Let X be a complete G -metric space (X, G) and $\{T_n\}$ be a sequence of self mappings on X . Assume that there exist three sequences (a_n) , (b_n) and (c_n) of elements of X such that*

$$\begin{aligned} G(T_i^p x, T_j^p y, T_k^p z) &\leq ({}_k \Delta_{i,j})G(x, y, z) + ({}_k \Theta_{i,j})[G(T_i^p x, x, x) + G(y, T_j^p y, y) + G(z, z, T_k^p z)] \\ &\quad + ({}_k \Lambda_{i,j})[G(T_i^p x, y, z) + G(x, T_j^p y, z) + G(x, y, T_k^p z)], \end{aligned} \quad (3.2)$$

for all $x, y, z \in X$ with $0 \leq {}_k \Delta_{i,j} + 3({}_k \Theta_{i,j}) + 4({}_k \Lambda_{i,j}) < 1/2$, $i, j, k = 1, 2, \dots$, some positive integer p , where ${}_k \Delta_{i,j} = G(a_i, a_j, a_k)$, ${}_k \Theta_{i,j} = G(b_i, b_j, b_k)$ and ${}_k \Lambda_{i,j} = G(c_i, c_j, c_k)$. If the sequence (r_i) where

$$r_i = \left[\frac{[({}_{i+2} \Delta_{i,i+1}) + 2({}_{i+2} \Theta_{i,i+1}) + 3({}_{i+2} \Lambda_{i,i+1})]}{1 - ({}_{i+2} \Theta_{i,i+1}) - ({}_{i+2} \Lambda_{i,i+1})} \right]$$

is a non-increasing λ -sequence of \mathbb{R}^+ endowed with the \max^6 metric, then $\{T_n\}$ have a unique common fixed point in X . Moreover, any fixed point of T_i is a fixed point of T_j for $i \neq j$.

The next result, corollary of Theorem 3.1, corresponds to the result presented by Abbas [1, Theorem 2.1].

Corollary 3.3. *Let X be a complete G -metric space (X, G) , f, g, h mappings on X . Assume that there exist three positive reals a, b, c such that*

$$\begin{aligned} G(fx, gy, hz) &\leq aG(x, y, z) + b[G(fx, x, x) + G(y, gy, y) + G(z, z, hz)] \\ &\quad + c[G(fx, y, z) + G(x, gy, z) + G(x, y, hz)], \end{aligned} \quad (3.3)$$

for all $x, y, z \in X$ with $0 \leq a + 3b + 4c < 1$. Then f, g, h have a unique common fixed point in X . Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. In Theorem 3.1, take $T_1 = f, T_2 = g, T_3 = h$. Also set

$${}_3 \Delta_{1,2} = a, \quad {}_3 \Theta_{1,2} = b, \quad {}_3 \Lambda_{1,2} = c.$$

Hence, we have:

⁶The max metric m refers to $m(x, y) = \max\{x, y\}$

$$0 \leq a + 3b + 4c < 1/2 \implies 0 \leq a + 3b + 4c < 1$$

$$\iff 0 \leq r_i = r = \left[\frac{a + 2b + 3c}{1 - b - c} \right] < 1.$$

The sequence $r_i = r$ is constant, so in Definition 1.8, if we choose $\lambda = \frac{1}{2}$ and $n(\lambda) = 1$, it is clear that $\sum_{i=1}^{\infty} r_i$ is an α -series. Indeed, since

$$\left[\frac{a + 2b + 3c}{1 - b - c} \right] < a + 3b + 4c < \frac{1}{2},$$

therefore, for any $L \geq n(\lambda) + 1 = 1 + 1 = 2$,

$$\sum_{i=1}^{L-1} r_i = \sum_{i=1}^{L-1} r < \frac{1}{2}(L-1) \leq \frac{1}{2}L.$$

□

We conclude this manuscript with the following result, whose proof is straightforward, following the steps of the proofs of the earliest results.

Theorem 3.4. *Let X be a complete G -metric space (X, G) and $\{T_n\}$ be a sequence of self mappings on X . Assume that there exist three sequences (a_n) , (b_n) and (c_n) of elements of X such that*

$$F[G(T_i^p x, T_j^p y, T_k^p z)] \leq F[({}_k\Delta_{i,j})G(x, y, z) + ({}_k\Theta_{i,j})[G(T_i^p x, x, x) + G(y, T_j^p y, y) + G(z, z, T_k^p z)] + ({}_k\Lambda_{i,j})[G(T_i^p x, y, z) + G(x, T_j^p y, z) + G(x, y, T_k^p z)]], \quad (3.4)$$

for all $x, y, z \in X$ with $0 \leq ({}_k\Delta_{i,j})^s + 3({}_k\Theta_{i,j})^s + 4({}_k\Lambda_{i,j})^s < 1/2$, $i, j, k = 1, 2, \dots$, some positive integer p and some $F \in \Phi$, homogeneous with degree s , where ${}_k\Delta_{i,j} = G(a_i, a_j, a_k)$, ${}_k\Theta_{i,j} = G(b_i, b_j, b_k)$ and ${}_k\Lambda_{i,j} = G(c_i, c_j, c_k)$. If the sequence (r_i) where

$$r_i = \left[\frac{[({}_{i+2}\Delta_{i,i+1})^s + 2({}_{i+2}\Theta_{i,i+1})^s + 3({}_{i+2}\Lambda_{i,i+1})^s]}{1 - ({}_{i+2}\Theta_{i,i+1})^s - ({}_{i+2}\Lambda_{i,i+1})^s} \right]$$

is a non-increasing λ -sequence of \mathbb{R}^+ endowed with the max⁷ metric, then $\{T_n\}$ have a unique common fixed point in X . Moreover, any fixed point of T_i is a fixed point of T_j for $i \neq j$.

In addition to the examples provided by Abbas and Vetro, illustrations of all the above results can be read in [2, Example 2.5] and [3, Example 2.8].

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests regarding the publication of this article.

⁷The max metric m refers to $m(x, y) = \max\{x, y\}$

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